

A deformation of the Curtright action

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We present a deformation of the action principle for a free tensor field of mixed symmetry $(2,1)$ –the Curtright action, a dual formulation of five-dimensional linearized gravity. It is constructed as the dual theory of the Einstein-Hilbert action linearized around a de Sitter background, and its derivation relies on the use of the two-potential formalism as an intermediate step. The resulting action principle is spatially non-local and space-time covariance is not manifest, thus overcoming previous no-go results.

I. INTRODUCTION

The construction of interacting gauge theories involving fields of mixed symmetry remains largely an open subject of research: although free theories can be constructed solely on the basis of the gauge principle [1]–[3], much is yet to be understood concerning interactions. There are two main areas of research where this type of tensor fields appear, and therefore the problem of deformations is relevant. One is the construction of interacting higher spin gauge theories, in relation with the mixed symmetry representations found in the infinite tower of higher spin massive excitations of the string theory spectrum. The other, which constitutes the subject of the present work, is the study of gravitational (or, more generally, higher spin) duality and hidden symmetries of gravitational theories.

There is a significant amount of indications [4]–[11] in support of the conjecture [12]–[16] that some extended Kac-Moody algebra underlies, as a larger symmetry of the unreduced theory, the well-established hidden symmetries of gravitational theories that appear upon toroidal compactification [17]–[19]. Within this picture, all the bosonic fields and their Hodge duals enter algebraically on an equal footing and are expected to be related among themselves by symmetry transformations of a highly non-trivial form. This would hold in particular for the graviton, with the peculiarity of being the only field in the bosonic sector of eleven dimensional supergravity whose Hodge dual, represented by a two-column $(D-3, 1)$ Young diagram, is described by a mixed symmetry tensor. Nevertheless, it is precisely this mixed symmetry character of the dual graviton the origin of the difficulties encountered in completing the proof of the conjecture. A particularly delicate impasse one meets in this regard is found in [20]: a no-go result that forbids, under the hypotheses of locality and manifest space-time covariance, deformations of the free action for a tensor field of $(2,1)$ Young type, originally proposed by Curtright as a dual description of linearized gravity in five dimensions [1]. These difficulties may be conceptually interpreted as a consequence of the lack of a notion of diffeomorphism covariance for mixed symmetry tensors.

The question we address here is whether it is possible to construct, by making use of the prepotential formal-

ism [21], a non-local theory, the dual of five-dimensional Einstein-Hilbert action linearized around some curved background metric, that one could regard as a deformation of the Curtright action –namely, in the flat limit one should recover the Curtright action and its gauge symmetries. We find that the answer is affirmative, at least, when the linearization is performed around a de Sitter background. The reason that justifies our focus on de Sitter space-time, as well as the election of the prepotential formalism, deserves some words of explanation. One first notices that de Sitter space-time admits a spatially flat slicing in an appropriate coordinate frame. This observation is crucial in order to extend the $SO(2)$ electric-magnetic duality invariance of the four-dimensional Pauli-Fierz theory [22] to the case where a positive cosmological constant is present [23]. In higher dimensions, on the other hand, it has been studied how to relate the Pauli-Fierz action to the dual theory using the prepotential formulation as an intermediate step [24]. Thus, it seems opportune to investigate the prepotential formulation of the five-dimensional Einstein-Hilbert action linearized around de Sitter space-time –expressed in flat slicing coordinates– with the purpose of constructing the corresponding deformation of the Curtright theory. We find that, working in a convenient gauge in which the linearized constraints take the same form as they do in the absence of a cosmological constant, one is able to construct such a deformation from the prepotential formulation of the action principle.

The rest of the article is organized as follows. In Section II we first describe the linearization of the Einstein-Hilbert action around a de Sitter background in any number of dimensions of the space-time manifold, as well as the canonical form of the theory; then we focus on five dimensions and solve the constraints in terms of two prepotentials, which turn out to retain the mixed symmetries found when the linearization is performed around a Minkowski background. After realizing in Section III that upon a particular gauge choice the constraints and the action take essentially the same form as in a Minkowski background, we construct the dual, non-local theory by introducing a pair of mixed symmetry canonical variables using the inversion formulae already known from the flat case. In Section IV we outline our conclusions and discuss the relevance of our result.

II. PREPOTENTIAL FORMULATION

A. Linearized action

In the presence of a cosmological constant, the ADM form [25]-[26] of the Einstein-Hilbert action principle for a metric $g_{\mu\nu}$ defined on a space-time manifold of arbitrary dimension D reads

$$S = \int d^D x \left[\pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N_i \mathcal{H}^i \right], \quad (\text{II.1})$$

with

$$\mathcal{H} = -g^{1/2}(R - 2\Lambda) - g^{-1} \left(\frac{\pi^2}{D-2} - g_{ik} g_{jl} \pi^{ij} \pi^{kl} \right) \quad (\text{II.2})$$

and

$$\mathcal{H}^i = -2\pi^{ij}{}_{|j}. \quad (\text{II.3})$$

The functions $N = (-g^{00})^{-1/2}$ and $N_i = g_{0i}$ denote respectively the lapse and the shift, whereas g stands for the determinant of the induced metric in $D-1$ dimensions, $g = \det(g_{ij})$.

In an arbitrary dimension D , the de Sitter line element expressed in spatially flat slicing coordinates takes the form

$$ds^2 = -dt^2 + f^2(t) \delta_{ij} dx^i dx^j \quad (\text{II.4})$$

with the definitions $f(t) = e^{kt}$ and $k = \sqrt{\frac{2\Lambda}{(D-1)(D-2)}}$. The linearization around this de Sitter background is achieved by setting

$$g_{ij} = \bar{g}_{ij} + h_{ij}, \quad \pi^{ij} = \bar{\pi}^{ij} + p^{ij} \\ N = 1 + n, \quad N_i = n_i, \quad (\text{II.5})$$

where the barred quantities are determined from (II.4):

$$\bar{g}_{ij} = f^2(t) \delta_{ij} \\ \bar{\pi}^{ij} = \sqrt{\bar{g}} (\bar{g}^{ij} \bar{K} - \bar{K}^{ij}) = -(D-2) k f^{D-3} \delta^{ij}.$$

By expanding (II.1) up to second order terms in the graviton variables one finds the linearized version of the action,

$$S = \int d^D x \left[p^{ij} \dot{h}_{ij} - H - nC - n_i C^i \right] \quad (\text{II.6})$$

with the Hamiltonian density

$$H = f^{-D+5} p_{ij} p^{ij} - \frac{f^{-D+5}}{D-2} p^2 - 2(D-3) k p_{ij} h^{ij} + k h p \\ + f^{D-7} \left[\frac{1}{4} \partial^i h^{jk} \partial_i h_{jk} - \frac{1}{4} \partial_i h \partial^i h + \frac{1}{2} \partial^i h \partial^j h_{ij} \right. \\ \left. - \frac{1}{2} \partial_i h^{ij} \partial^k h_{kj} \right] - k^2 f^{D-5} \frac{(D-2)(-2D+6)}{4} h_{ij} h^{ij} \quad (\text{II.7})$$

and the linearized constraints

$$C \equiv f^{D-5} (\Delta h - \partial_i \partial_j h^{ij}) + 2k p f^2 \\ + f^{D-3} k^2 h (D-2)(D-3) = 0 \quad (\text{II.8})$$

$$C^i \equiv -2\partial_j p^{ij} \\ + (D-2) f^{D-5} k (2\partial_k h^{ik} - \partial^i h) = 0 \quad (\text{II.9})$$

The latter are first-class and, as such, generate the gauge transformations of the canonical variables via the Poisson brackets

$$\delta h_{ij} = \{h_{ij}, \int d^{D-1} x (-\xi C + \xi_m C^m)\} \\ = \partial_i \xi_j + \partial_j \xi_i - 2k f^2 \delta_{ij} \xi \\ \delta p_{ij} = \{p_{ij}, \int d^{D-1} x (-\xi C + \xi_m C^m)\} \\ = f^{D-5} (-\partial^i \partial^j \xi + \delta_{ij} \Delta \xi) \\ + (D-2)(D-3) k^2 f^2 \delta_{ij} \xi \\ + (D-2) k (\partial_i \xi_j + \partial_j \xi_i - \partial_m \xi^m \delta_{ij}). \quad (\text{II.10})$$

One should notice that the Hamiltonian and the kinetic term in (II.6) are not gauge invariant (up to a total derivative) by themselves, contrarily to the case $\Lambda = 0$. Instead, the variation of the former compensates the variation of the latter. The reason for this is the explicit dependence of the Hamiltonian and the constraints on the time-like variable t . This forces the introduction of explicit time derivatives in the equations expressing the conservation in time of the constraints [27], which otherwise would only involve brackets:¹

$$\dot{C} = \frac{\partial C}{\partial t} + \{C, \int d^{D-1} y H\} = 0 \quad (\text{II.11})$$

$$\dot{C}^m = \frac{\partial C^m}{\partial t} + \{C^m, \int d^{D-1} y H\} = 0. \quad (\text{II.12})$$

B. Solving the constraints

In order to solve the constraints, it is most useful to perform the canonical transformation

$$h_{ij} \mapsto \hat{h}_{ij} = h_{ij} \\ p^{ij} \mapsto \hat{p}^{ij} = p^{ij} - \frac{D-2}{2} k f^{D-5} (2h^{ij} - \delta^{ij} h). \quad (\text{II.13})$$

¹ As described in [27], in order to eliminate explicit time derivatives one may promote the time-like coordinate t to a canonical variable by a reparametrization $t = t(\tau)$. The phase space is then enlarged to contain both t and its canonical momentum.

This transformation may be derived from the generating functional

$$F[h_{ij}, \hat{p}^{ij}] = \int d^{D-1}x \left[\hat{p}^{ij} h_{ij} + \frac{(D-2)}{2} k f^{D-5} (h_{ij} h^{ij} - \frac{1}{2} h^2) \right] \quad (\text{II.14})$$

depending on the ‘old’ field h_{ij} and the ‘new’ conjugate momentum \hat{p}^{ij} , for it reproduces² (II.13):

$$\begin{aligned} \frac{\delta F}{\delta h^{ij}} &\equiv p_{ij} = \hat{p}_{ij} + \frac{D-2}{2} k f^{D-5} (2h_{ij} - \delta_{ij} h) \\ \frac{\delta F}{\delta \hat{p}^{ij}} &\equiv \hat{h}_{ij} = h_{ij}. \end{aligned}$$

The action principle (II.6) reduces then to

$$S[\hat{p}^{ij}, h_{ij}, n, n_i] = \int d^D x \left[\hat{p}^{ij} \dot{h}_{ij} - H - nC - n_i C^i \right], \quad (\text{II.15})$$

where now the Hamiltonian and the constraints read, respectively,

$$\begin{aligned} H &= f^{-D+5} \hat{p}_{ij} \hat{p}^{ij} - \frac{f^{-D+5}}{D-2} \hat{p}^2 + 2k \hat{p}_{ij} h^{ij} \\ &+ f^{D-7} \left[\frac{1}{4} \partial^i h^{jk} \partial_i h_{jk} - \frac{1}{4} \partial_i h \partial^i h + \frac{1}{2} \partial^i h \partial^j h_{ij} \right. \\ &\left. - \frac{1}{2} \partial_i h^{ij} \partial^k h_{kj} \right] \end{aligned} \quad (\text{II.16})$$

and

$$C = f^{D-5} (-\partial^i \partial^j h_{ij} + \Delta h) + 2k f^2 \hat{p} \quad (\text{II.17})$$

$$C^i = -2\partial_j \hat{p}^{ij}. \quad (\text{II.18})$$

We notice a dependence on the dimension of space-time D in the scalar constraint, whereas the vector constraint does not depend on D at all. The new canonical variable \hat{p}^{ij} transforms as

$$\delta \hat{p}^{ij} = f^{D-5} (-\partial^i \partial^j \xi + \delta^{ij} \Delta \xi). \quad (\text{II.19})$$

In the sequel we shall focus on a five-dimensional space-time, where the dual graviton is described by the Curtright field, i.e., a mixed symmetry tensor field of Young type (2,1). We shall solve the constraints in terms of prepotentials and write down the corresponding action principle.

The solution of the vector constraint (II.18) has been found in [24] by making use of the cohomological results for N -complexes and the generalized Poincaré lemma for rectangular Young diagrams [29]:

$$\hat{p}^{ij} = \epsilon^{iklm} \epsilon^{jnpq} \partial_k \partial_n P_{lmnp}. \quad (\text{II.20})$$

The prepotential P_{abcd} is a mixed symmetry tensor of Young type (2,2). The transformations

$$\begin{aligned} \delta P_{abcd} &= 2\chi_{cd[b,a]} + 2\chi_{ab[d,c]} + \frac{1}{4} [\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}] \xi, \\ \chi_{abc} &= -\chi_{bac}, \chi_{[abc]} = 0 \end{aligned} \quad (\text{II.21})$$

reflect the ambiguity in the choice of P_{abcd} : they leave \hat{p}^{ij} invariant up to the gauge transformation $\delta \hat{p}^{ij} = -\partial^i \partial^j \xi + \delta^{ij} \Delta \xi$. Thus, the parameter ξ in (II.21) produces the gauge transformations of \hat{p}^{ij} , whereas χ_{abc} defines an internal invariance.

On the other hand, the expansion of the product of the totally antisymmetric tensors in (II.20) yields

$$\begin{aligned} \hat{p}^{ij} &= \delta^{ij} (2\Delta P_{ab}^{ab} - 4\partial_a \partial_b P_{ab}^{ab}) - 2\partial^i \partial^j P_{ab}^{ab} \\ &+ 4\partial^i \partial_a P^{jba}_b + 4\partial^j \partial_a P^{iba}_b \\ &- 4\Delta P^{iaj}_a + 4\partial_a \partial_b P^{iajb} \end{aligned} \quad (\text{II.22})$$

so one gets from its trace and (II.13)

$$p = -3kh + 2\Delta P_{ab}^{ab} - 4\partial_a \partial_b P_{ab}^{ab}. \quad (\text{II.23})$$

Substitution in the scalar constraint (II.17) produces

$$\Delta h - \partial_i \partial_j h^{ij} + 4f^2 k \Delta P_{ab}^{ab} - 8f^2 k \partial_i \partial_j P^{imj}_m = 0. \quad (\text{II.24})$$

In order to solve the previous equation we shall decompose the prepotential P_{ijkl} as follows:

$$P_{abcd} = Q_{abcd} + \frac{1}{12} [\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}] P_{mn}^{mn} \quad (\text{II.25})$$

with Q_{ijkl} a (2,2) tensor whose double trace vanishes. According to (II.21), the term carrying the double trace of (II.25) is a gauge transformation of parameter $\xi = P_{mn}^{mn}/3$. This transformation also affects h_{ij} through (II.10) but not the form of the scalar constraint, owing to its gauge invariance. Equation (II.24) may then be written in the form

$$\Delta j - \partial^i \partial^j j_{ij} + 4k f^2 \Delta Q_{ab}^{ab} - 8k f^2 \partial^i \partial^j Q_{ikj}^k = 0 \quad (\text{II.26})$$

with $h_{ij} = j_{ij} - \frac{2}{3} k f^2 \delta_{ij} P_{mn}^{mn} + \partial_i u_j + \partial_j u_i$. One may choose u^i in such a way that $j = 0$ so the scalar constraint reduces to

$$\partial^i \partial^j (j_{ij} + 8k f^2 Q_{ikj}^k) = 0. \quad (\text{II.27})$$

² For a discussion on generating functions in classical mechanics and how they are used to define the ‘old’ and ‘new’ sets of canonical variables, see for instance [28].

This equation can be solved exactly in the same manner as the Hamiltonian constraint in the case $\Lambda = 0$ [24]:

$$j_{ij} + 8kf^2 Q_{ikj}{}^k = \partial^k \epsilon_{ikab} \phi_j^{ab} + \partial^k \epsilon_{jkab} \phi_i^{ab}. \quad (\text{II.28})$$

The traceless condition on the left-hand side of (II.28) guarantees that the prepotential ϕ_{ijk} is a mixed symmetry tensor transforming in the (2,1) irreducible representation. Our final expression for h_{ij} reads

$$\begin{aligned} h_{ij} &= \partial^k \epsilon_{ikab} \phi_j^{ab} + \partial^k \epsilon_{jkab} \phi_i^{ab} - 8kf^2 Q_{ikj}{}^k \\ &\quad + \partial_i u_j + \partial_j u_i - \frac{2}{3} k f^2 \delta_{ij} P^{mn}{}_{mn} \\ &= \partial^k \epsilon_{ikab} \phi_j^{ab} + \partial^k \epsilon_{jkab} \phi_i^{ab} + \partial_i u_j + \partial_j u_i \\ &\quad - 8kf^2 P_{ikj}{}^k + \frac{4}{3} k f^2 \delta_{ij} P^{mn}{}_{mn}. \end{aligned} \quad (\text{II.29})$$

Now we shall determine the invariances of the prepotential ϕ_{ijk} . They are defined by the equation

$$\begin{aligned} \delta h_{ij} &= \partial_i \xi_j + \partial_j \xi_i - 2kf^2 \delta_{ij} \xi \\ &= \partial^l \epsilon_{ilab} \delta \phi_j^{ab} + \partial^l \epsilon_{jl ab} \delta \phi_i^{ab} + \partial_i \delta u_j + \partial_j \delta u_i \\ &\quad - 8kf^2 \delta P_{ikj}{}^k + \frac{4}{3} k f^2 \delta_{ij} \delta P^{mn}{}_{mn}. \end{aligned} \quad (\text{II.30})$$

Substituting for δP_{abcd} according to (II.21) one gets

$$\begin{aligned} \partial^l \epsilon_{ilab} \delta \phi_j^{ab} + \partial^l \epsilon_{jl ab} \delta \phi_i^{ab} &= \partial_i (\xi_j - \delta u_j + 8kf^2 \chi_{jb}{}^b) \\ &\quad + \partial_j (\xi_i - \delta u_i + 8kf^2 \chi_{ib}{}^b) \\ &\quad - 8kf^2 \partial^l \chi_{jli} - 8kf^2 \partial^l \chi_{ili} \\ &\quad - \frac{16}{3} k f^2 \delta_{ij} \partial^l \chi_{lb}{}^b. \end{aligned} \quad (\text{II.31})$$

It is useful to dualize in the antisymmetric pair of indices of χ_{abc} (and then project onto its (2,1) symmetry)

$$\chi_{abc} = 2\epsilon_{abxy} \tilde{\chi}_{yc} + \epsilon_{cbxy} \tilde{\chi}_{xya} - \epsilon_{caxy} \tilde{\chi}_{xyb}$$

so (II.31) takes the form

$$\begin{aligned} &\partial^l \epsilon_{il}{}^{ab} (\delta \phi_{abj} + 16kf^2 (\tilde{\chi}_{jba} + \tilde{\chi}_{abj})) \\ &+ \partial^l \epsilon_{jl}{}^{ab} (\delta \phi_{abi} + 16kf^2 (\tilde{\chi}_{iba} + \tilde{\chi}_{abi})) \\ &= \partial_i (\xi_j - \delta u_j + 16kf^2 \epsilon_{jbox} \tilde{\chi}^{xyb}) \\ &+ \partial_j (\xi_i - \delta u_i + 16kf^2 \epsilon_{ibxy} \tilde{\chi}^{xyb}). \end{aligned} \quad (\text{II.32})$$

From the previous expression one finally deduces

$$\begin{aligned} \delta \phi_{abc} &= \partial_a S_{bc} - \partial_b S_{ac} + \partial_a A_{bc} - \partial_b A_{ac} + 2\partial_c A_{ba} \\ &\quad + B_{[a} \delta_{b]c} - 16kf^2 (\tilde{\chi}_{cab} + \tilde{\chi}_{abc}) \end{aligned} \quad (\text{II.33})$$

and

$$\delta u_i = \xi_i + 16kf^2 \epsilon_{ibxy} \tilde{\chi}^{bxy} - 2\partial^l \epsilon_{il}{}^{ab} A_{ba}. \quad (\text{II.34})$$

We observe the appearance of terms in $\tilde{\chi}_{ijk}$, corresponding to an invariance of the prepotential P_{ijkl} , as a part of the transformations of the prepotential ϕ_{ijk} as well: the introduction of a cosmological constant mixes the invariance parameters of the prepotentials in this particular manner. The fact that this phenomenon does not occur in four dimensions may be attributed to its especially symmetric character.

C. Action in the prepotential formalism

Having solved the constraints, we can now write the action principle in terms of the prepotentials by direct substitution in (II.15). This yields

$$\begin{aligned} S[\phi_{ijk}, P_{abcd}] &= \int dt d^4x \left[2\epsilon^{imab} \epsilon^{jncd} \epsilon_{ilxy} \partial_m \partial_n P_{abcd} \partial^l \dot{\phi}^{xy}{}_j \right. \\ &\quad + \frac{32}{3} k \dot{P} \partial_a \partial_b P^{ab} - 8k \dot{P}_{ij} \partial_a \partial_b P^{iajb} \\ &\quad + 8kf^2 \epsilon^{jlab} \partial^i \partial_l \phi_{ab}{}^k \partial_i P_{jk} - 8kf^2 \epsilon_{jlab} \partial_i \partial^l \phi^{abi} \partial_k P^{kj} \\ &\quad + \frac{72}{9} k^2 f^2 \partial_j P \partial^j P + 32k^2 f^2 \partial^i P_{ik} \partial_j P^{jk} \\ &\quad - 16f^2 k^2 \partial_i P_{jk} \partial^i P^{jk} - \frac{64}{3} k^2 f^2 \partial_i P \partial_j P^{ij} \\ &\quad - \left(f^{-4} (R_{ij}[P] R^{ij}[P] - \frac{7}{27} R^2[P]) \right. \\ &\quad \left. \left. + f^2 (2E_{ijk}[\phi] E^{ijk}[\phi] - \frac{3}{2} E_i[\phi] E^i[\phi]) \right) \right]. \end{aligned} \quad (\text{II.35})$$

Here R_{ij} and E_{ijk} are defined as in [24]: they correspond to the respective contractions of the tensors

$$R_{ijklmn} = 18\partial_{[i} P_{jk][lm,n]} \quad (\text{II.36})$$

and

$$E_{ijkmn} = 6\partial_{[n} \partial_{i[T_{jk]m]} \quad (\text{II.37})$$

regarded as curvatures for the prepotentials. We see that, in addition to the terms that appear when $\Lambda = 0$, there are new terms proportional to k (a power of Λ) with two derivatives of the prepotentials. Contrarily to the situation in four dimensions, one can not get rid of these terms even after a redefinition by a power of f in the prepotentials (for instance, the terms proportional to k with time derivatives can not be written as a total time derivative). This may again be interpreted as an indication of a special character of the four dimensional case: the complications appearing in higher dimensions happen to vanish, which ultimately gives rise to the $SO(2)$ symmetry acting on the prepotentials.

One may now wonder whether it is possible to derive (II.35) from some suitable deformation of the Curtright

action once the corresponding constraints are solved. Since one has no prior knowledge of any such deformation, this may in fact be regarded as a definition: the deformed theory, dual to the Einstein-Hilbert action linearized around a de Sitter background (II.1), must be such that it takes the form (II.35) after the resolution of the constraints. However, this definition does not provide any indications about how to construct the deformed action. For such purpose, it seems more convenient to restrict ourselves to a particular gauge in which the terms in (II.35) proportional to k are not present: the Hamiltonian would then take the same form as in the case $\Lambda = 0$ (up to factors of f) and thus one could introduce a dual pair of canonical variables following the steps described in [24]. We shall see in next section that such a gauge does exist, and the gauge-fixed deformation of the Curtright action will be derived. This suffices in order to show the novel qualitative feature of the dual action principle: its non-local character.

III. DEFORMED ACTION

The aim of this section is to show how the prepotential formulation previously developed turns out to be useful in the construction of a dual theory satisfying the properties we expect for a deformation of the Curtright action. In order to simplify the analysis, we shall work in the gauge where $\hat{p} = 0$ (we see from (II.13) that this condition is satisfied, in particular, in the transverse-traceless gauge of [26]). In this gauge, the scalar constraint (II.17) takes the same form as in the case with no cosmological constant: this leads to the vanishing of the terms proportional to k and k^2 in (II.35), and simplifies the construction of the deformed action.

Thus we shall set

$$\begin{aligned}\hat{p}^{ij} &= a^{ij} + \delta\hat{p}^{ij} = \hat{a}^{ij} - \partial^i\partial^j\xi + \delta^{ij}\Delta\xi \\ h_{ij} &= b_{ij} - 2kf^2\xi\end{aligned}\quad (\text{III.1})$$

and $a = 0$. Clearly this can be achieved through the gauge choice $\xi = \frac{1}{3}\Delta^{-1}\hat{p}$. Now the constraints read as in the case with no cosmological constant [24]

$$\partial_j a^{ij} = 0 \quad (\text{III.2})$$

$$\Delta b - \partial^i\partial^j b_{ij} = 0 \quad (\text{III.3})$$

so they may be solved as follows

$$a^{ij} = f^{-2}\partial^k\partial^l\epsilon_{ikab}\epsilon_{jlcd}P^{abcd} \quad (\text{III.4})$$

$$b_{ij} = f^2(\partial^l\epsilon_{ilab}\phi^{ab}_j + \partial^l\epsilon_{jlab}\phi^{ab}_i) + \partial_i u_j + \partial_j u_i. \quad (\text{III.5})$$

We note that u_i is such that $b = 2\partial^m u_m$, and it may actually be dropped for it has the form of a gauge transformation. The prepotentials have been redefined by powers of f for future convenience.

By substituting in the action (II.15) one gets

$$\begin{aligned}S[P_{ijkl}, \phi_{abc}] &= \\ &\int dt d^4x \left[2\partial_m\partial_k\epsilon^{imnp}\epsilon^{jksl}P_{npst}\partial^l\epsilon_{ilab}\dot{\phi}^{ab}_j \right. \\ &\quad \left. - \left(f^{-4}(R_{ij}[P]R^{ij}[P] - \frac{7}{27}R^2[P]) \right. \right. \\ &\quad \left. \left. + f^2(2E^{ijk}[\phi]E_{ijk}[\phi] - \frac{3}{2}E_i[\phi]E^i[\phi]) \right) \right].\end{aligned}\quad (\text{III.6})$$

We observe that, because of our gauge choice (III.1), the terms proportional to k and k^2 in (II.35) are not present any longer, and thus the action has the same form as in the absence of a cosmological constant (up to some powers of f in the Hamiltonian). In this regard, the redefinition of the prepotentials implicit in (III.4) and (III.5) by powers of f is crucial, for it permits the cancellation of the contributions derived from the term $2kp^{ij}h_{ij}$ in the Hamiltonian (II.16) –owing to a term of opposite sign produced by the time derivative in the kinetic term.

In order to construct the dual theory one now introduces the canonical pair of dual variables defined as follows [24]:

$$\begin{aligned}\hat{t}^{ijk} &= -\frac{2}{3}f^{-2}\partial_l \left[2\epsilon^{klab}P_{ab}{}^{ij} + \epsilon^{ilab}P_{ab}{}^{kj} - \epsilon^{jlab}P_{ab}{}^{ki} \right] \\ \hat{\pi}_{ijk} &= f^2\epsilon_{ijmn}\epsilon_{krst}\partial^m\partial^r\phi^{stn}.\end{aligned}\quad (\text{III.7})$$

The action

$$S[\hat{t}_{ijk}, \hat{\pi}_{ijk}, m_j, m_{ij}] = \int d^5x \left[\hat{\pi}^{ijk}\dot{\hat{t}}_{ijk} - \mathcal{H} - m_j\Gamma^j - m_{jk}\Gamma^{jk} \right] \quad (\text{III.8})$$

reproduces the form of (III.6), with the Hamiltonian density

$$\begin{aligned}\mathcal{H} &= -2k\hat{\pi}^{ijk}\hat{t}_{ijk} + \frac{1}{2}\partial_i\hat{t}_{jkl}\partial^i\hat{t}^{jkl} + \partial_i\hat{t}_{jkl}\partial^j\hat{t}^{kil} \\ &\quad - \frac{1}{2}\partial^k\hat{t}_{ijk}\partial_l\hat{t}^{ijl} + \frac{1}{2}\hat{\pi}_{ijk}\hat{\pi}^{ijk} - \frac{1}{2}\hat{\pi}_i{}^{ji}\hat{\pi}^k{}_{jk}\end{aligned}\quad (\text{III.9})$$

and the constraints

$$\Gamma^j = \partial_i\partial_k\hat{t}^{ijk} \quad (\text{III.10})$$

$$\Gamma^{ij} = -2\partial_k(\hat{\pi}^{ijk} + \hat{\pi}^{kji}). \quad (\text{III.11})$$

The latter need to be imposed so (III.7) holds, and generate the usual gauge transformations

$$\begin{aligned}\delta t_{ijk} &= 2\partial_{[i}u_{j]k} + 2\partial_{[i}v_{j]k} - 2\partial_k v_{ij}, \\ \delta\pi^{ijk} &= \frac{1}{2}(\partial^j\partial^k\xi^i - \partial^i\partial^k\xi^j) \\ &\quad + \frac{1}{2}(\delta^{jk}(\partial^i\partial_m\xi^m - \Delta\xi^i) - \delta^{ik}(\partial^j\partial_m\xi^m - \Delta\xi^j))\end{aligned}\quad (\text{III.12})$$

leaving (III.8) invariant, with $u_{ij} = u_{ji}$, $v_{ij} = -v_{ji}$.

The action (III.8) may be regarded as the dual of the standard action (II.15) –written in terms of the ‘new’ variables (b_{ij}, a^{ij}) . In order to obtain an action dual to the original variational principle (II.6) –in terms of the ‘old’, graviton canonical variables (h_{ij}, p^{ij}) – we should find a way to invert the action of the canonical transformation (II.13) in the dual picture. This is achieved by writing the generating functional (II.14) in terms of the variables \hat{t}_{ijk} and $\hat{\pi}^{ijk}$ through the inversion formulae [24]

$$\phi_{ijk}[\hat{\pi}] = -\frac{1}{2}\Delta^{-1}\hat{\pi}_{ijk} \quad (\text{III.13})$$

and

$$\begin{aligned} P_{abcd}[\hat{t}] = & \frac{1}{8} \left[\epsilon_{abij} \partial^i \Delta^{-1} \hat{t}_{cd}^j + \epsilon_{cdij} \partial^i \Delta^{-1} \hat{t}_{ab}^j \right] - \\ & -\frac{1}{24} \left[\epsilon_{abij} \partial^i \Delta^{-1} \hat{t}_{cd}^j + \epsilon_{cdij} \partial^i \Delta^{-1} \hat{t}_{ab}^j \right. \\ & + \epsilon_{caij} \partial^i \Delta^{-1} \hat{t}_{bd}^j + \epsilon_{adij} \partial^i \Delta^{-1} \hat{t}_{bc}^j \\ & \left. + \epsilon_{bcij} \partial^i \Delta^{-1} \hat{t}_{ad}^j + \epsilon_{bdij} \partial^i \Delta^{-1} \hat{t}_{ca}^j \right]. \end{aligned} \quad (\text{III.14})$$

These expressions are suitable for our gauge choice, for they imply $a = b = 0$. In addition to the condition $\hat{t}_{ij}^j = 0$ (implicit in (III.7)) we can further impose that $\hat{\pi}_{ij}^j = 0$: it is consistent with the inversion formula (III.13), in the sense that the trace of ϕ_{ijk} is pure gauge [24], and simplifies the computations.³

Since the canonical transformation (II.13) leaves h_{ij} invariant, it is natural to expect in the dual theory the action of the canonical transformation on $\hat{\pi}_{ijk}$ to be the identity map. Therefore we shall set $\hat{\pi}_{ijk} = \pi_{ijk}$. The generating functional takes the form

$$F[\pi^{ijk}, \hat{t}_{ijk}] = \int d^4x [-\hat{t}^{ijk} \pi_{ijk} - 3k \pi^{ijk} \Delta^{-1} \pi_{ijk}]$$

where we have dropped a boundary term. We observe that, when expressed in terms of the dual variables, the generating functional depends on the ‘old’ conjugate momentum π^{ijk} and the ‘new’ field \hat{t}_{ijk} , so the relevant relations are now

$$t_{ijk} = -\frac{\delta F}{\delta \pi^{ijk}}, \quad \hat{\pi}^{ijk} = -\frac{\delta F}{\delta \hat{t}_{ijk}}. \quad (\text{III.15})$$

³ We should remind that the gauge parameter that implements a gauge condition on π^{ijk} (similarly, on t_{ijk}) corresponds to an internal invariance of h_{ij} (π_{ij}), so the imposition of a gauge condition for the mixed symmetry canonical variables does not modify the gauge parameters of the canonical variables in the standard theory (and vice versa).

The equation for the conjugate momentum yields $\pi^{ijk} = \hat{\pi}^{ijk}$, in agreement with our previous guess. For the Curtright field, one finds the spatially non-local expression

$$t_{ijk} = -\frac{\delta F}{\delta \pi^{ijk}} = \hat{t}_{ijk} + 6k \Delta^{-1} \pi_{ijk} \quad (\text{III.16})$$

The action (III.8) is now expressed in terms of the pair (t_{ijk}, π^{ijk}) :

$$\begin{aligned} S[t_{ijk}, \pi^{ijk}, m_i, m_{ij}] = & \int dt d^4x [\pi^{ijk} \dot{t}_{ijk} - \mathcal{H} \\ & - m_i \Gamma^i - m_{ij} \Gamma^{ij}] \end{aligned} \quad (\text{III.17})$$

We have dropped a total time derivative originating from the kinetic term and recast the Hamiltonian density as the sum

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_\Lambda \quad (\text{III.18})$$

with

$$\begin{aligned} \mathcal{H}_0 = & \frac{1}{2} \partial_i t_{jkl} \partial^i t^{jkl} + \partial_i t_{jkl} \partial^j t^{kjl} - \frac{1}{2} \partial^k t_{ijk} \partial_l t^{ijl} \\ & + \frac{1}{2} \pi_{ijk} \pi^{ijk} - \frac{1}{2} \pi_i^{ji} \pi_{jk}^k \end{aligned} \quad (\text{III.19})$$

the contribution from the free theory and

$$\mathcal{H}_\Lambda = 4k \pi^{ijk} t_{ijk} - 6k^2 \pi^{ijk} \Delta^{-1} \pi_{ijk} \quad (\text{III.20})$$

the term carrying the deformation. As in (II.6), one distinguishes two contributions to the deformation: one linear and other quadratic in k , with the novelty here of the non-local character of the latter. The constraints read

$$\Gamma^j = \partial_i \partial_k (t^{ijk} - 6k \Delta^{-1} \pi_{ijk} - \delta_{ijk}) \quad (\text{III.21})$$

$$\Gamma^{ij} = -2 \partial_k (\pi^{ijk} + \pi^{kji}). \quad (\text{III.22})$$

The action principle (III.17) may then be regarded as a gauge-fixed deformation of the Curtright action in its Hamiltonian form [24]. Its most prominent feature consists in a spatially non-local character, reflected in the term proportional to k^2 of the deformed Hamiltonian, in consonance with the main result in [20]. The undeformed theory is recovered in the limit $k \rightarrow 0$. To the best of our knowledge, this is the first instance of a deformation of the Curtright action in the literature.

IV. CONCLUSIONS AND COMMENTS

We have constructed an action principle, dual to the linearized version of five-dimensional gravity around a de Sitter background, in terms of the mixed symmetry canonical variables associated to the dual graviton. This construction can be naturally regarded as a deformation of the Curtright action corresponding to the introduction of a positive cosmological constant in the Pauli-Fierz

theory and relies on the resolution of the corresponding constraints in terms of two prepotentials, which turn out to possess the same mixed symmetries as in the case with no cosmological constant. Though the resolution of these constraints can be carried out with no need to specify any gauge condition, the construction of the dual action is particularly simplified upon a specific gauge choice, where the constraints and the Hamiltonian –after performing a suitable canonical transformation defining new variables $(\hat{h}_{ij}, \hat{\pi}^{ij})$ – take essentially the same form as they do in a Minkowski background (up to time-dependent factors). Consequently, the prepotential formulation of the action principle also takes the same form as in the case $\Lambda = 0$ (again, up to time-dependent factors), and this allows us to introduce a canonical pair of dual variables by pertinent inversion formulae, imposing the usual Curtright constraints on them. These, however, are the dual version of the ‘new’, canonically transformed variables $(\hat{h}_{ij}, \hat{\pi}^{ij})$, not of the ‘old’, original ones (h_{ij}, π^{ij}) . After reversing the canonical transformation in the dual picture, one obtains a spatially non-local action principle. This is the main qualitative feature of our construction, sufficient to circumvent no-go results based on the hypotheses of locality and manifest Lorentz invariance.

Since we have specified a particular gauge in the derivation of our result, one may wish to deal with the general situation and derive an ungauged deformed action. In this ungauged version of the dual theory, one expects both the Hamiltonian and the constraints to include new terms, in such a way that the ungauged prepotential action (II.35) is recovered when the constraints are solved. As we have observed, gauge transformations of the standard canonical variables (h_{ij}, π^{ij}) do not affect the dual, mixed symmetry canonical variables (and viceversa), so there is no way to determine these additional terms in the dual theory from the knowledge of the gauge parameters in the standard picture. Nevertheless, one can guess some of their properties by simple arguments. For instance, one would expect a deformation in the constraint for $\hat{\pi}^{ijk}$ by terms depending solely on \hat{t}_{ijk} , and no deformation in the constraint for \hat{t}_{ijk} , thus inverting the situation encountered in the standard picture (a deformation in the constraint for h_{ij} depending only on \hat{p}^{ij} and no deformation in the constraint for \hat{p}^{ij}). Moreover, one should be able to recast the deformation of the constraint for $\hat{\pi}^{ijk}$ in the form of a gauge transformation $\delta\hat{t}_{ijk}$ (so it can actually be gauged away), with the parameter depending on \hat{t}_{ijk} , and its vanishing should imply $\hat{p} = 0$. A strategy to determine the exact form of these additional terms in the ungauged dual action is to consider all possible contributions that satisfy the aforementioned requirements and then substitute in the action in order to adjust their relative coefficients, in such a way that (II.35) is recovered in the end. It would be of interest to know the exact form of these additional terms, and see whether they bring about any qualitatively new features.

One should also emphasize that the deformed, non-local action we have derived corresponds merely to the introduction of a cosmological constant in the Pauli-Fierz picture, and not to the inclusion of higher order terms in the perturbative expansion of the Einstein-Hilbert action. On the other hand, one may well wonder about the existence of deformations of the Curtright theory corresponding to linearization of the Einstein-Hilbert action around other space-time backgrounds, such as anti de Sitter or, more generally, conformally flat metrics. If such a deformation should be obtained by the method we have described, this would require not only the feasibility of a prepotential resolution of the corresponding linearized constraints (as yet, a subject to be investigated also in four dimensions), but also our ability to recast them, by some change of variables, in the same form as in the flat case: in this way one could establish the Curtright constraints on the dual side and simply introduce the dual pair of canonical variables by the already known inversion formulae, reversing the change of variables on the dual side afterwards.

Apart from these technical issues, our result also rises more conceptual questions, such as the interpretation of a cosmological constant in the dual theory, or the properties that space and time should possess in such an alternative formulation –where the concept of constant curvature in the geometric theory gets translated into spatial non-locality⁴. The answer to these questions is certainly far from obvious, and it would constitute a part of the challenging enterprise of interpreting higher dimensional, (pseudo) Riemannian geometry in a dual, non-geometric formulation where the basic objects possess a mixed symmetry character. Since the necessity of such a complicated description of gravity seems hard to justify a priori –especially when we dispose of the elegant, generally covariant description that General Relativity provides– we can only hope that a further understanding of gravitational duality may be relevant in order to gain definitive insight into the, yet conjectural and rather mysterious, extended Kac-Moody algebra structure.

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⁴ A possibility to be considered is whether one can reestablish either locality or manifest space-time covariance by the introduction of auxiliary fields.

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